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# Some Erdös–Feldheim Type Theorems on Mean Convergence of Lagrange Interpolation

A. K. VARMA* AND P. VÉRTESI*

*Department of Mathematics, University of Florida, Gainesville, Florida 32611*

*Submitted by G.-C. Rota*

## 1. INTRODUCTION AND PRELIMINARY RESULTS

Let $w(x)$ be a weight function in $[-1, 1]$ and let $\{p_n(w; x)\}$ denote the corresponding sequence of orthonormal polynomials. Denote

$$-1 < x_{nn}(w) < x_{n-1,n}(w) < \cdots < x_{1n} < 1 \tag{1.1}$$

the zeros of $p_n(w, x)$ and, sometimes omitting the superfluous notations,

$$L_n(f, w, x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x) \tag{1.2}$$

the Lagrange interpolatory polynomials of degree $\leqslant n-1$, where $f \in C$ ($=f$ is continuous on $[-1, 1]$) and

$$l_{kn}(x) = l_{kn}(w; x) = p_n(x)/(x - x_{kn})\, p_n'(x_{kn}). \tag{1.3}$$

A classical theorem of Erdös and Turán [3] states that

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n(f, w, x) - f(x)]^2 w(x)\, dx = 0 \tag{1.4}$$

if $f \in C$. As it was proved by Askey [1, 2] the power 2 generally cannot be replaced by $2 + \varepsilon$. But considering the Tchebyshev weight

$$w^{(-1/2,-1/2)}(x) = (1 - x^2)^{-1/2} \tag{1.5}$$

Erdös and Feldheim [4] showed

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THEOREM 1.1 (Erdös, Feldheim). *By the above notations*

$$\lim_{n\to\infty}\int_{-1}^{1} |L_n(f, w^{(-1/2,-1/2)}, x) - f(x)|^p \frac{dx}{\sqrt{1-x^2}} = 0 \qquad (1.6)$$

*if $f \in C$ whenever $p > 0$ is a fixed number.*

## 2. New Results

2.1

The aim of this paper is to prove results analogous to (1.6) using different node system and involving endpoints of the interval too. Let $x_{0n}(w) = -x_{n+1,n}(w) = 1$. If $Q_{n+2}(f, w, x)$ stands for the Lagrange interpolatory polynomials of degree $\leqslant n+1$ based on $\{x_{kn}(w)\}$, $k = 0, 1, \ldots, n+1$, for $f \in C$, further $w^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$ is a Jacobi weight ($\alpha, \beta > -1$), then using the extended Tchebyshev nodes of the second kind we state

THEOREM 2.1. *For any fixed $p > 0$*

$$\lim_{n\to\infty}\int_{-1}^{1} |Q_{n+2}(f, w^{(1/2,1/2)}, x) - f(x)|^p \frac{dx}{\sqrt{1-x^2}} = 0 \qquad (2.2)$$

*if $f \in C$.*

Using a different proof we can obtain

THEOREM 2.2. *For any fixed $p > 0$*

$$\lim_{n\to\infty}\int_{-1}^{1} |Q_{n+2}(f, w^{(3/2,3/2)}, x) - f(x)|^p \frac{dx}{\sqrt{1-x^2}} = 0 \qquad (2.3)$$

*if $f \in C$.*

2.2

If we consider $L_n(f, w, x)$ instead of $Q_{n+2}(f, w, x)$, then the relations analogous to (2.2) (or (2.3)) hold only if $p < 1$ (or $p < \frac{1}{2}$) (see Nevai [8] and Vértesi [12]).

2.3

Using different weight functions Feldheim [5] proved that for certain functions $f_1 \in C$ and $f_2 \in C$

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n(f_1, w^{(1/2,1/2)}, x) - f(x)]^4 \sqrt{1-x^2} = \infty,$$

and

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n(f_2, w^{(1/2,1/2)}, x) - f(x)]^2 \, dx = \infty$$

(for general setting, see [8]).

2.4

By the method of the corresponding proofs similar results can be obtained using the Lagrange interpolation based on the nodes $x_{kn}(w^{(-1/2,1/2)})$, $k = 1, 2, \ldots, n+1$ (see (2.2)) or using the Lagrange interpolation based on $x_{kn}(w^{(-1/2,3/2)})$ $k = 1, 2, \ldots, n+1$ (see (2.3)) for any $p > 0$, so we omit the details.

## 3. Proof of Theorem 2.1

3.1

First we prove a lemma which is analogous to the result of Fejér (see Rivlin [9, p. 26, Formula 1.49]). Let

$$-1 = x_{n+1,n} < x_{n,n} < \cdots < x_{1n} < x_{0n} = 1 \tag{3.1}$$

be the zeros of $(1 - x^2)\, u_n(x)$, where

$$u_n(x) = \sin(n+1)\,\theta/\sin\theta, \qquad x = \cos\theta, \tag{3.2}$$

$$\phi_{0,n}(x) = (1+x)\, u_n(x)/2\, u_n(1), \qquad \phi_{n+1,n}(x) = (1-x)\, u_n(x)/2 u_n(-1), \tag{3.3}$$

$$\phi_{kn}(x) = (1-x^2)\, u_n(x)/(1-x_{kn}^2)(x - x_{kn})\, u_n'(x_{kn}), \qquad k = 1, 2, \ldots, n, \tag{3.4}$$

the fundamental polynomials of Lagrange interpolation based on the points system (3.1), and throughout in this section we denote

$$Q_{n+2}(f, x) = \sum_{k=0}^{n+1} f(x_{kn})\, \phi_{k,n}(x) \tag{3.5}$$

for $Q_{n+2}(f, w^{(1/2,1/2)}, x)$.

LEMMA 3.1. *For $-1 \leqslant x \leqslant 1$ we have*

$$\sum_{k=1}^{n} \phi_{kn}^2(x) \leqslant 2, \qquad |\phi_{kn}(x)| \leqslant \sqrt{2}, \qquad k = 1, 2, \ldots, n, \tag{3.6}$$

*and*

$$\sum_{k=1}^{n} |\phi_{kn}^l(x)| \leqslant C, \qquad l = 3, 4, \ldots, \tag{3.7}$$

*where C is independent of n.*

*Proof.* Let $T_n(x) = \cos n\theta$, $x = \cos\theta$, then from the known relation

$$(1 - x^2)\, u_n'(x) = x u_n(x) - (n+1)\, T_{n+1}(x) \tag{3.8}$$

we obtain

$$(1 - x_{kn}^2)\, u_n'(x_{kn}) = (-1)^{k+1}(n+1).$$

Using (3.4) we obtain

$$\sum_{k=1}^{n} \phi_{kn}^2(x) = [(1-x^2)^2 u_n^2(x)/(n+1)^2] \sum_{k=1}^{n} (1/(x - x_{kn})^2).$$

Next, using the known identity

$$u_n^2(x) \sum_{k=1}^{n} (1/(x - x_{kn})^2) = (u_n'(x))^2 - u_n(x)\, u_n''(x)$$

(P. Turán [10, p. 94])(3.8) and

$$(1 - x^2)\, u_n''(x) - 3x u_n'(x) + n(n+2)\, u_n(x) = 0$$

we obtain

$$\sum_{k=1}^{n} \phi_{kn}^2(x) = \frac{1}{(n+1)^2} [n(n+2) + T_{n+1}^2(x) + x(n+1)\, T_{n+1}(x)\, u_n(x) \\ - 2x^2 u_n^2(x)] \leqslant 2,$$

This proves (3.6). Equation (3.7) follows from (3.6).

3.2

In the work of Erdös and Feldheim [4] following result played an important role. Let $l_{\nu n}(x) \equiv l_\nu(x)$, $\nu = 1, 2, \ldots, n$ be the fundamental polynomials of Lagrange interpolation based on the zeros of $T_n(x)$ as abscissas. Then if $k$ is even,

$$\int_{-1}^{1} l_{\nu_1}(x)\, l_{\nu_2}(x) \cdots l_{\nu_k}(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0,$$

where $\nu_1, \nu_2, ..., \nu_k$ are distinct integers between 1 and $n$. For the extended Tchebychev nodes of the second kind the corresponding result is given by the following.

LEMMA 3.2. *Let $k$ be even and $\nu_1, \nu_2, ..., \nu_k$ be distinct integers between* 1 *and $n$. Then we have* $(k = 2l)$

$$\left| \int_{-1}^{1} \phi_{\nu_1}(x)\, \phi_{\nu_2}(x) \cdots \phi_{\nu_k}(x) \frac{1}{\sqrt{1-x^2}}\, dx \right| = \frac{\Gamma(l+(1/2))\,\Gamma(1/2)}{(n+1)^{2l}\Gamma(l+1)}. \tag{3.9}$$

*Proof.* Since

$$\sin^{2l-1}\theta = \sum_{p=1}^{l} \alpha_{2p-1,l} \sin(2p-1)\,\theta,$$

where we note that

$$\alpha_{1,l} = \frac{2}{\pi} \int_0^{\pi} \sin^{2l}\theta d\theta = \frac{2}{\pi} \frac{\Gamma(l+(1/2))\,\Gamma(1/2)}{\Gamma(l+1)}, \tag{3.10}$$

from above it follows that

$$\sin^{2l-1}(n+1)\,\theta = \sum_{p=1}^{l} \alpha_{2p-1,l} \sin \lambda_{p,n}\theta, \qquad \lambda_{p,n} = (2p-1)(n+1).$$

Also

$$\begin{aligned}(1-x^2)^l(u_n(x))^{2l-1} &= \sin\theta \sin^{2l-1}(n+1)\,\theta \\ &= \frac{1}{2} \sum_{p=1}^{l} \alpha_{2p-1,l}(\cos(\lambda_{p,n}-1)\,\theta - \cos(\lambda_{p,n}+1)\,\theta) \\ &= \frac{1}{2} \alpha_{1,l} \cos n\theta + \sum_{i>n}^{(2l-1)(n+1)+1} \mu_i \cos i\theta.\end{aligned}$$

It is also easy to see that

$$(1-x^2)^l u_n(x)/(x-x_{\nu_1})(x-x_{\nu_2}) \cdots (x-x_{\nu_{2l}}) = (-1)^l u_n(x) + q_{n-1}(x),$$

where $q_{n-1}(x)$ is a polynomial of degree $\leqslant n-1$. On using these ideas together with orthogonality property of Tchebychev polynomials with weight function $(1-x^2)^{-1/2}$ we obtain

$$\begin{aligned}\left|\int_{-1}^{1} \phi_{v_1}(x)\,\phi_{v_2}(x)\cdots\phi_{v_{2l}}(x)\frac{1}{\sqrt{1-x^2}}\,dx\right| &= \frac{1}{2}\frac{\alpha_{1,2l}}{(n+1)^{2l}}\left|\int_{-1}^{1} u_n(x)\,T_n(x)\frac{1}{\sqrt{1-x^2}}\,dx\right| \\ &= (\pi/2)(\alpha_{1,2l}/(n+1)^{2l}).\end{aligned}$$

On using (3.10) we obtain (3.9).

3.3

In Lemma 3.3 we prove Theorem 2.1 for $p = 2$.

LEMMA 3.3. *Let $f \in C$, then*

$$\lim_{n\to\infty} \int_{-1}^{1} [Q_{n+2}(f, x) - f(x)]^2 \frac{1}{\sqrt{1-x^2}}\,dx = 0. \tag{3.11}$$

*Proof.* From the known theorem of Gopengauz [6] it follows that there exist a sequence of polynomials $P_{n+1}(x)$ of degree $\leqslant n+1$ such that

$$|\psi_n(x)| \equiv |f(x) - P_{n+1}(x)| \leqslant C\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right), \tag{3.12}$$

where $w(f, \delta)$ denotes the modulus of continuity of $f$. Now, on using (3.5), (3.6), (3.9), and (3.12) we obtain

$$\begin{aligned}\int_{-1}^{1} Q_{n+2}^2(\psi_n, x)\frac{1}{\sqrt{1-x^2}}\,dx &= \int_{-1}^{1} \sum_{k=1}^{n} \psi_n^2(x_{kn})\,\phi_{kn}^2(x)\frac{1}{\sqrt{1-x^2}}\,dx \\ &\quad + \sum\sum_{k\neq j} \psi_n(x_{kn})\,\psi_n(x_{jn})\int_{-1}^{1} \phi_{kn}(x)\,\phi_{jn}(x)\frac{dx}{\sqrt{1-x^2}} \\ &\leqslant C \max_{-1\leqslant x\leqslant 1} \psi_n^2(x) = o(1).\end{aligned} \tag{3.13}$$

From (3.12), (3.13)

$$\begin{aligned}\int_{-1}^{1} [Q_{n+2}(f, x) - f(x)]^2 \frac{1}{\sqrt{1-x^2}}\,dx &\leqslant 2\int_{-1}^{1} Q_{n+2}^2(\psi_n, x)\frac{1}{\sqrt{1-x^2}}\,dx + 2\int_{-1}^{1} \psi_n^2(x)\frac{1}{\sqrt{1-x^2}}\,dx = o(1).\end{aligned} \tag{3.14}$$

This proves (3.11). From (3.13) we note that

$$\int_{-1}^{1} |Q_{n+2}[\psi_n, x]| \frac{dx}{\sqrt{1-x^2}} = o(1). \tag{3.15}$$

3.4

For the proof of Theorem 2.1 in the general case we again follow the method of Erdös and Feldheim [4]. It is enough to prove for even values of $p$ only. To illustrate the method we limit for the case $p = 4$. For arbitrary fixed even $p$ the proof is similar.

Thus we need to show that for $f \in C$

$$\lim_{n\to\infty} \int_{-1}^{1} [Q_{n+2}(f, x) - f(x)]^4 \frac{1}{\sqrt{1-x^2}} dx = 0. \tag{3.16}$$

Proceeding as in (3.14) it is easy to see that (3.16) is valid provided we can show that

$$\lim_{n\to\infty} \int_{-1}^{1} Q_{n+2}^4(\psi_n, x) \frac{1}{\sqrt{1-x^2}} dx = 0 \tag{3.17}$$

and

$$\lim_{n\to\infty} \int_{-1}^{1} |Q_{n+2}^3(\psi_n, x)| \frac{1}{\sqrt{1-x^2}} dx = 0. \tag{3.18}$$

Equation (3.18) follows from (3.17) so we now prove (3.17). From (3.5) and (3.12) we obtain

$$\begin{aligned} Q_{n+2}^4(\psi_n, x) = {} & \sum_{k=1}^{n} \psi_n^4(x_{kn}) \phi_{kn}^4(x) \\ & + \sum_{k\neq j}\sum \psi_n^3(x_{kn}) \psi_n(x_{in}) \phi_{kn}^3(x) \phi_{jn}(x) \\ & + \sum_{k\neq j\neq i}\sum\sum \psi_n^2(x_{kn}) \psi_n(x_{jn}) \psi_n(x_{in}) \phi_{kn}^2(x) \phi_{jn}(x) \phi_{in}(x) \\ & + \sum_{k\neq j\neq i\neq s}\sum\sum\sum \psi_n(x_{kn}) \psi_n(x_{jn}) \psi_n(x_{in}) \psi_n(x_{sn}) \phi_{kn}(x) \\ & \times \phi_{jn}(x) \phi_{in}(x) \phi_{sn}(x) \\ & + \sum_{k\neq j}\sum \psi_n^2(x_{kn}) \psi_n^2(x_{jn}) \phi_{kn}^2(x) \phi_{jn}^2(x) \\ = {} & J_1(x) + J_2(x) + J_3(x) + J_4(x) + J_5(x). \end{aligned} \tag{3.19}$$

From (3.7) and (3.12) we obtain

$$\int_{-1}^{1} J_1(x) \frac{1}{\sqrt{1-x^2}}\, dx = o(1). \tag{3.20}$$

Next, note that

$$J_2(x) = \left( \sum_{k=1}^{n} \psi_n^3(x_{kn})\, \phi_{kn}^3(x) \right) \left( \sum_{k=1}^{n} \psi_n(x_{kn})\, \phi_{kn}(x) \right) - J_1(x).$$

Using Lemma 3.1, (3.12), (3.15), and (3.20) we obtain

$$\int_{-1}^{1} |J_2(x)| \frac{1}{\sqrt{1-x^2}}\, dx = o(1). \tag{3.21}$$

From (3.9) and (3.12) we have

$$\int_{-1}^{1} |J_4(x)| \frac{1}{\sqrt{1-x^2}}\, dx = o(1). \tag{3.22}$$

For the estimation of $\int_{-1}^{1} |J_3(x)|\, 1/\sqrt{1-x^2}\, dx$ we note that

$$J_3(x) = \left( \sum_{k=1}^{n} \psi_n^2(x_{kn})\, \phi_{kn}^2(x) \right) \left[ Q_{n+2}^2(\psi_n, x) - \sum_{j=1}^{n} \psi_n^2(x_{jn})\, \phi_{jn}^2(x) \right] - J_2(x).$$

Thus, using (3.13), (3.20), Lemma 1, and (3.12) we obtain

$$\int_{-1}^{1} |J_3(x)| \frac{1}{\sqrt{1-x^2}}\, dx = o(1). \tag{3.23}$$

Last,

$$J_5(x) = \left( \sum_{k=1}^{n} \psi_n^2(x_{kn})\, \phi_{kn}^2(x) \right)^2 - J_1(x)$$

using Lemma 1, (3.12), and (3.20) we obtain

$$\int_{-1}^{1} |J_5(x)| \frac{1}{\sqrt{1-x^2}}\, dx = o(1). \tag{3.24}$$

Combining (3.19)–(3.24) we obtain (3.17) which leads to (3.16).

## 4. Proof of Theorem 2.2

### 4.1

Let $x_{kn} = \cos\theta_{kn}$, $1 \leqslant k \leqslant n$, be the nodes of the process $L_n$ based on the roots of $T_n(x) = \cos n\theta$; $y_{kn} = \cos\eta_{kn}$, $1 \leqslant k \leqslant n$, $y_1 \equiv 1$, $y_n \equiv -1$, are the nodes of the process $Q_n$ based on the roots of $(1-x^2)\, P_{n-2}^{(3/2,3/2)}(x)$. Here and later $P_n^{(\alpha,\beta)}(x)$ means the Jacobi polynomials of degree $= n$ with the usual normalization (see [11, Eq. 4.1]). By [11, Eq. 2.6] and $u_n(x) - u_{n-2}(x) = 2T_n(x)$ we get

$$\begin{aligned} c_n(1-x^2)\, P_{n-2}^{(3/2,3/2)}(x) &= (n-1)\, u_n(x) - (n+1)\, u_{n-2}(x) \\ &= 2(n-1)\, T_n(x) - 2u_{n-2}(x). \end{aligned}$$

It is easy to see that $c_n \sim n^{3/2}$ (for "$\sim$") see [11, Eq. 1.1]). So by a proper $d_n \sim n^{1/2}$ we can define $p_n(x)$ as follows:

$$p_n(x) = d_n(1-x^2)\, P_{n-2}^{(3/2,3/2)}(x) = T_n(x) - u_{n-2}(x)/(n-1). \tag{4.1}$$

If $\phi_{kn}$ are the fundamental functions based on the roots of $p_n(x)$ and $y_{j(n)}$ is the nearest root to $x$, then with $K = \min(k, n-k+1)$ and $K + J = \min(k+j, 2n-k-j+1)$ it is easy to see that

$$|\phi_{kn}(x)| = O(1)\, K/(K+J)(|k-j|+1), \qquad 1 \leqslant k \leqslant n, \tag{4.2}$$

from where

$$\sum_{k=1}^{n} |\phi_{kn}(x)| = O(1n\, n); \tag{4.3}$$

here and later $x \in [-1, 1]$ and the "0" means absolute (positive) constant. (Indeed, if, e.g., $2 \leqslant k \leqslant n-1$, we can write

$$\begin{aligned} \phi_k(x) &= \frac{1-x^2}{1-y_k^2} \frac{P_n^{(3/2,3/2)}(x)}{P_n^{(3/2,3/2)}(y_k)(x-y_k)} \\ &= 0(1) \frac{\sin^2\theta\, \theta^{-2}\, n^{-1/2}}{\sin^2\eta_k\, K^{-3}\, n^{7/2} \sin((\theta+\eta_k)/2) \sin((\theta-\eta_k)/2)} \\ &= O(1)\, K/(K+J)(|k-j|+1) \qquad \text{if} \quad k \neq j \end{aligned}$$

(see [11, Eq. 7.32(3), (8.9.2)] and the formula $\eta_{k+1} - \eta_k \sim n^{-1}$, $k = 1, 2, \ldots, n-1$, from [7]). The remaining cases can be similarly settled using (3.3)–(3.4)). As it is well known, the analogous formulae hold for $l_k(x) = T_n(x)[T_n'(x_k)(x-x_k)]^{-1}$.

4.2

If $f \in C$ and $|f(x)| \leqslant 1$, we state with a suitable absolute constant $M$ that

$$S = \int_I \left| \sum_{k=M}^{n-M} f(y_k)\, \phi_k(x) \right|^p \frac{dx}{\sqrt{1-x^2}} = O(1), \qquad p > 0, \tag{4.4}$$

where with $[z_{k+1}, z] = [\cos(\theta_{k+1} - n^{-4/3}), \cos(\theta_k + n^{-4/3})]$, $a_n = [\sqrt{n}]$ and $b_n = n - a_n$

$$I_n = \bigcup_{k=a_n}^{b_n} [z_{k+1}, z_k] \qquad \text{and} \qquad \overline{I_n} = [-1, 1] \backslash I_n. \tag{4.5}$$

To prove (4.4) first we verify

$$\phi_k(x) = \frac{T_n(x)}{T_n'(x_{kn})(x - x_{kn})} \left[ 1 + \frac{O(1)}{K} + \frac{O(1)}{n^{1/6}} \right]$$

$$\text{if} \quad x \in I_n, \quad M \leqslant k \leqslant n - M. \tag{4.6}$$

Indeed, by (4.1)

$$\begin{aligned} p_n(x) &= T_n(x) - (u_{n-2}(x)/(n-1)) = T_n(x) + O(n^{-1/2}) \\ &= T_n(x)[1 + O(n^{-1/6})] \end{aligned}$$

if $x \in I_n$; again by (4.1)

$$\begin{aligned} p_n'(y_k) &= T_n'(y_k) - \frac{u_{n-2}'(y_k)}{n-1} = T_n'(y_k)[1 + O(K^{-1})] \\ &= T_n'(y_k)[1 + O(K^{-1})]^{-1} \end{aligned}$$

if $M$ is big enough and $M \leqslant k \leqslant n - M$.

Finally, using that

$$|\theta_{kn} - \eta_{kn}| = O(1/Kn), \qquad 1 \leqslant k \leqslant n, \tag{4.7}$$

(see the argument of [11, Eq. 8.9 (1), especially (8.9.3)] and [11, Eq. (4.21.7)] or see [13, Eq. 4.8.1]),

$$\begin{aligned} x - y_k &= (x - x_k)[1 + O(n^{-1/6})] \\ &= (x - x_k)[1 + O(n^{-1/6})]^{-1} \qquad \text{if} \quad x \in I_n, \quad n \geqslant n_0. \end{aligned}$$

By these formulae we obtain (4.6).

4.3

Now, using (4.6) and $|a+b|^p \leqslant c_p(|a|^p+|b|^p)$ ([14, Sect. 5.2(10)]), we get

$$\begin{aligned} S = O(1) \Bigg[ \int_{I_n} \left| \sum_{k=M}^{n-M} f(y_k)\, l_k(x) \right|^p \frac{dx}{\sqrt{1-x^2}} \\ + \int_{I_n} \left| \sum_{k=M}^{n-M} f(y_k) \frac{l_k(x)}{K} \right|^p \frac{dx}{\sqrt{1-x^2}} \\ + \frac{1}{n^{1/6}} \int_{I_n} \left| \sum_{k=M}^{n-M} f(y_k)\, l_k(x) \right|^p \frac{dx}{\sqrt{1-x^2}} \Bigg] \\ \equiv O(1)[S_1+S_2+S_3]. \end{aligned}$$

First we state that $S_1 = O(1)$. Indeed, we write

$$\begin{aligned} S_4 &= \int_{-1}^{1} \left| \sum_{k=1}^{n} f(y_k)\, l_k(x) \right|^p \frac{dx}{\sqrt{1-x^2}} \\ &= O(1) \Bigg[ \int_{-1}^{1} \Bigg| \sum_{\substack{1 \leqslant k < M \\ n-M < k \leqslant n}} \cdots \Bigg|^p \frac{dx}{\sqrt{1-x^2}} \\ &\quad + \int_{I_n} \left| \sum_{k=M}^{n-M} \cdots \right|^p \frac{dx}{\sqrt{1-x^2}} + \int_{\bar{I}_n} \left| \sum_{k=M}^{n-M} \cdots \right|^p \frac{dx}{\sqrt{1-x^2}} \Bigg] \\ &\equiv O(1)[S_5+S_1+S_6]. \end{aligned}$$

By $|l_k(x)| = O(1)$, $1 \leqslant k \leqslant n$, we obtain $S_5 = O(1)$; moreover, $S_6 = O(1)$ considering that $\sum_{k=1}^{n} |l_k(x)| = O(\ln n)$ and $|\bar{I}_n| = O(n^{-1/3})$. Moreover, $S_4 = O(1)$ because using the proof of Theorem 1.1, it is easy to see that

$$\int_{-1}^{1} \left| \sum_{k=1}^{n} g_{kn}\, l_{kn}(x) \right|^p \frac{dx}{\sqrt{1-x^2}} = O(1), \qquad p > 0$$

if $|g_{kn}| \leqslant 1$ for any $k$ and $n$. Using these facts we have that $S_1 = O(1)$.

4.4

To obtain $S_2 = O(1)$ we remark that $\sum_{k=1}^{n} |l_k(x)\, K^{-1}| = O(1)$ (see (4.2)); finally by $\sum_{k=1}^{n} |l_k(x)| = O(\ln n)$ (see (4.3)) we get that $S_3 = O(1)$, which means $S = O(1)$, as we stated.

4.5

The remaining part is easy. Indeed, let us write

$$\begin{aligned} S_9 &= \int_{-1}^{1} \left| \sum_{k=1}^{n} f(y_k)\,\phi_k(x) \right|^p \frac{dx}{\sqrt{1-x^2}} \\ &= O(1) \left[ \int_{-1}^{1} \left| \sum_{\substack{1\leqslant k<M \\ n-M<k\leqslant n}} \cdots \right|^p \frac{dx}{\sqrt{1-x^2}} \right. \\ &\quad \left. + \int_{I_n} \left| \sum_{k=M}^{n-M} \cdots \right|^p \frac{dx}{\sqrt{1-x^2}} + \int_{\bar{I}_n} \left| \sum_{k=M}^{n-M} \cdots \right|^p \frac{dx}{\sqrt{1-x^2}} \right] \\ &\equiv O(1)(S_7 + S_8 + S_9). \end{aligned}$$

Here $S_7 = O(1)$ because $|\phi_k(x)| = O(1)$, further $S_8 = O(1)$ by $|\bar{I}_n| = O(n^{-1/3})$ and (4.3). So $S_9 = O(1)$, which actually proves the theorem.

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